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Analogue of Black Strings in Yang–Mills Gauge Theory

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The classical Yang–Mills equations are analyzed within the geometrical framework of an effective gravity theory. Exact analytical solutions are derived for the cylindrically symmetric configurations of the coupled gauge and isoscalar fields. It turns out that there is an infinite family of solutions parametrized by two real parameters, one of which determines the asymptotic behavior of fields near the symmetry axis and in infinity, while the second locates the singularity. These configurations have a simple pole at a finite value of the radial coordinate, and physically they represent "thick string"-like objects which possess the confinement properties. It is demonstrated that the particles with gauge charge cannot move classically and quantum mechanically out of the interior region. Such objects are thus direct analogues of the "black string" gravitational configurations reported recently in the literature.

1. INTRODUCTION

Interest in classical solutions of Yang–Mills gauge theory has been recently revived on the basis of correspondences between gauge models with internal symmetry group and gauge theories of gravity. There is a wide variety of such maps (Johnson, 1993; Freedman *et al*., 1993; Freedman and Khuri, 1994; Lunev, 1992, 1993a, c, 1994; Bauer *et al*., 1994; Bauer and Freedman, 1995; Mielke *et al*., 1994; Haagenson and Johnson, 1995; Radovanovic and Šijački, 1995), which permit the reformulation of Yang–Mills theory in terms of an "effective" gravitational Einstein or, in general, Einstein–Cartan theory. In particular, the use of such reformulations enabled a number of authors (Lunev, 1993b; Singleton, 1995) to find spherically symmetric solutions for the Yang–Mills equations which are analogous to black hole gravitational

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configurations. Quite recently the study of cylindrically symmetric gravitational fields has resulted in "black string" solutions (Lemos, 1994; Lemos and Zanchin, 1996) which possess a similar property providing confinement of a classical particle to the inner region under the cylindrical horizon. In the present paper we use the mapping technique to derive the analogous exact cylindrically symmetric solutions in Yang–Mills gauge theory.

To recall, in previous studies of the classical Yang–Mills theory attention was paid mainly to the regular solutions with finite energy or/and action [a good review of earlier work is given in Actor (1979)]. However it was proven (Swank *et al.*, 1975) that finite-energy gauge field configurations cannot form bound states with fermion particles and thus do not possess confining properties. It was proposed in Swank *et al.*, (1975) that infinite-energy configurations should be analyzed instead. The new spherically symmetric solutions (Lunev, 1993b; Singleton, 1995) are all singular and have infinite energy, which indeed yields confining properties analogous to that of a black hole. [As a matter of fact, these solutions are not really new; both Lunev (1993b) and Singleton (1995) rediscovered the results of two papers of Protogenov (1977, 1979); however their discussion of classical confinement is a new development; see also Sington and Yoshida (1995).]

There was already some interest in more general symmetries; in particular, Witten (1979) was the first to establish a direct correspondence between axially symmetric Einstein solutions and static, axially symmetric, self-dual Yang–Mills gauge fields. A more recent, partly overlapping discussion is presented in Singleton (1996). A cylindrically symmetric case was analyzed in (Mahajan and Valanju, 1987a, b), which was treated as a leading approximation for the toroidal localized configurations.

In the present paper we show that singular solutions with confining properties exist not only for spherical symmetry (Lunev, 1992, 1993a, c, 1994; Singleton, 1995), but also for a cylindrically symmetric case. The general interest in this case is motivated by studies of classical stringlike configurations, in particular in cosmology. When, however, periodic conditions are imposed on the symmetry axis coordinate, one finds approximate toroidal solutions. Compact toroidal configurations of gluonic and quark matter have attracted much attention (see, e.g., Mahajon and Valanju (1987a, b) and Robson (1980) and references therein). It is also worthwhile to mention that the study of the cylindrically symmetric case is often useful as a preliminary step in treating a more general axial symmetric problem. We use the mapping from $SU(2)$ Yang–Mills theory into effective Einstein–Cartan gravity for the formulation of the general problem and for the analysis of the properties of the solutions obtained. An infinite family of solutions is discovered which are labeled by two real parameters.

2. YANG–MILLS THEORY MAPPED INTO A GAUGE GRAVITY MODEL

Let us consider $SU(2)$ gauge theory on flat Minkowski space-time M. It will be convenient to use the coordinate-free formulation in terms of the exterior form language. As in Singleton (1995), we will study the model of coupled $SU(2)$ gauge field and a triplet of massless Higgs scalar fields ϕ^a, i.e., what is called the Yang–Mills–Higgs theory in the Prasad–Sommerfield limit. The Lagrangian is

$$L_{\text{YMH}} = -\frac{1}{2}\left(F^a \wedge {}^*F_a + D\phi^a \wedge {}^*D\phi_a\right) \tag{2.1}$$

where $*$ denotes the four-dimensional Hodge dual.

In order to describe the mapping from the internal gauge theory into an "effective" gravity form, we need to make a formal $(1 + 3)$ decomposition. Hence we will distinguish the global time coordinate $x^0 = t$ from the rest of the coordinates x^i, $i = 1, 2, 3$, which parametrize the flat Euclidean three-dimensional space $\underline{M}$ (so that $M = R \times \underline{M}$). The (pseudo)-Riemannian structure is introduced on M by the Minkowski metric g with the Lorentzian signature $(-, +, +, +)$, and in accordance with the product structure we write the line element as

$$ds^2 = -dt \otimes dt + \underline{ds}^2 = -dt \otimes dt + \underline{g}_{ij}\, dx^i \otimes dx^j \tag{2.2}$$

with the positive-definite 3-metric $\underline{ds}^2 = \underline{g}_{ij}\, dx^i \otimes dx^j$. The latter is flat: a coordinate transformation exists from x^i to x'^i in which $\underline{g}'_{ij} = \delta_{ij}$. Hereafter the underline denotes the three-dimensional quantities and structures on $\underline{M}$.

Developing the $(1 + 3)$-decomposition (see, e.g., Mielke, 1992) of the $SU(2)$ Yang–Mills theory defined on M, one writes the potential 1-form and the field strength 2-form as

$$A^a = dtA_0^a + \underline{A}^a, \qquad F^a = dt \wedge E^a + B^a \tag{2.3}$$

The 'electric' piece of the Yang–Mills field strength reads

$$E^a = \dot{\underline{A}}^a - \underline{D}A_0^a = D_t\underline{A}^a - \underline{d}A_0^a \tag{2.4}$$

with $\dot{\underline{A}} = \partial_0 \underline{A}$ as the time derivative, and the covariant derivatives defined by

$$D_t\xi^a := \dot{\xi}^a + \varepsilon^a_{bc}A_0^b\xi^c, \qquad \underline{D}\xi^a := \underline{d}\xi^a + \varepsilon^a_{bc}\underline{A}^b\xi^c \tag{2.5}$$

The 'magnetic' piece of the field strength is

$$B^a := \underline{F}^a = \underline{dA}^a + \frac{1}{2}\varepsilon^a_{bc}\underline{A}^b \wedge \underline{A}^c \tag{2.6}$$

The Lagrangian (2.1) is decomposed as

$$L_{\mathrm{YMH}} = \tfrac{1}{2} dt \wedge (-B^a \wedge {}^*B_a + E^a \wedge {}^*E_a + D_t\phi^a \wedge \underline{*} D_t\phi_a \\ - \underline{D}\phi^a \wedge \underline{*}\underline{D}\phi_a) \tag{2.7}$$

where $\underline{*}$ stands for *the 3-dimensional Hodge dual*, defined by the flat 3-metric $\underline{g}$.

We now transform the Yang–Mills theory into the form of an effective gauge gravity model. There are several ways to do this (Johnson, 1993; Freedman *et al.*, 1993; Freedman and Khuri, 1994; Lunev, 1992, 1993a, c, 1994, Bauer *et al.*, 1994, Bayer and Freedman, 1995. Mielke *et al.*, 1994, Haagensen and Johnson, 1995), but here we consider the mapping due to Lunev (1992, 1993a–c, 1994), which is defined when a fixed background (e.g., the Minkowskian) geometry is present. As a first step, we rewrite Yang–Mills theory in first-order form. This is straightforward after the introduction of two auxiliary fields, the 1-form Θ_a and the 2-form π_a (both transform covariantly under the action of the gauge group):

$$L_{\mathrm{YMH}} = dt \wedge \Bigg(- \Theta_a \wedge B^a + \frac{1}{2}\Theta_a \wedge \underline{*}\Theta^a \\ + \pi_a \wedge E^a - \frac{1}{2}\pi_a \wedge {}^*\pi^a + D_t\phi^a \wedge {}^*D_t\phi_a - \underline{D}\phi^a \wedge {}^*\underline{D}\phi_a \Bigg) \tag{2.8}$$

The basic mapping is formulated as follows: for every three-dimensional Yang–Mills configuration (A^a, Θ^a) we define *the three-dimensional Riemann–Cartan effective* geometry by

$$\underline{\Gamma}^{ab} = A^c \varepsilon_c^{\ ab}, \qquad \underline{\vartheta}^a = \Theta^a \tag{2.9}$$

Then the magnetic strength (2.6) is mapped into the three-dimensional curvature two-form $\underline{R}^{ab}$ and first term in the Lagrangian (2.8) is just the standard Hilbert–Einstein gravitational term,

$$-\Theta_a \wedge B^a = -\frac{1}{2}\eta_{ab} \wedge \underline{R}^{ab}$$

while the second term represents a sort of a generalized "cosmological constant." Except for $(\underline{\vartheta}^a, \underline{\Gamma}^{ab})$ all the rest of the variables (the Yang–Mills field time component A_0^a, the variable π^a, and ϕ^a) then should be treated as the effective matter described by the second line in (2.8). Let us derive the effective "gravitational" field equations. Independent variation of (2.8) with respect to $\underline{\vartheta}^a$ and $\underline{\Gamma}^{ab}$ yields, respectively,

$$\underline{R}^{ab} = \varepsilon^{abc}\, {}^{*}\underline{\vartheta}_c \tag{2.10}$$

$$\underline{T}^a = -\, D_t\pi^a - \varepsilon^a_{bc}\phi^b\underline{{}^{*}D}\phi^c \tag{2.11}$$

where $\underline{T}^a$ is the effective torsion two-form. Notice the absence of a "matter source" in the effective Einstein equation (2.10). As one notices [correcting the statements made in Lunev (1992, 1993a–c, 1994)], in the general case the effective spatial geometry is the Riemann–Cartan one: torsion (2.11) is nontrivial for nonstatic configurations, and both A_0 and Higgs scalars contribute to it. Variation of (2.8) with respect to the "matter" fields yields, respectively,

$$\underline{D}\pi^a = -\, \varepsilon^a_{bc}\phi^b\underline{{}^{*}}D_t\phi^c, \qquad E^a = \underline{{}^{*}}\, \pi^a \tag{2.12}$$

while the scalar multiplet ϕ^a satisfies the generalized Klein–Gordon field equation,

$$D_t\, {}^{*}D_t\phi^a - \underline{D{}^{*}D}\phi^a = 0 \tag{2.13}$$

3. CYLINDRICALLY SYMMETRIC CONFIGURATIONS

We are going to discuss the exact solutions with cylindrical symmetry. Hence we cover the 3-space $\underline{M}$ by the cylindrical coordinate system, $x^i = \{\rho, \theta, z\}$, with the background 3-metric in its standard form

$$\underline{ds}^2 = d\rho^2 + \rho^2\, d\theta^2 + dz^2 \tag{3.1}$$

We look for the static, cylindrically symmetric solutions of the Yang–Mills equations. In the effective gravity theory (2.11)–(2.10) we should search for the effective frame and connection. The most general cylindrically symmetric ansatz reads

$$\underline{\vartheta}^a = \begin{pmatrix} A(\rho)d\rho \\ B(\rho)d\theta \\ C(\rho)dz \end{pmatrix}, \qquad \underline{\Gamma}^b_a = \begin{pmatrix} 0 & Ud\theta & V\rho^{-1}dz \\ -Ud\theta & 0 & 0 \\ -V\rho^{-1}dz & 0 & 0 \end{pmatrix} \tag{3.2}$$

Hereafter we use the self-evident matrix notation. The functions $U = U(\rho)$, $V = V(\rho)$, A, B, and C determine the gauge field configuration. Extra ρ factors are introduced in (3.2) for later convenience. It seems worthwhile to mention that in the effective gravity framework it is quite straightforward to derive the symmetric ansatz for any field variable. In particular, (3.2) is suggested naturally by the Cartan structure equations. The torsion and curvature 2-forms for the Riemann–Cartan gauge fields (3.2) are as follows:

$$T^a = \begin{pmatrix} 0 \\ (B' + AU)d\rho \wedge d\theta \\ (C' + AV\rho^{-1})d\rho \wedge dz \end{pmatrix} \tag{3.3}$$

$$R_a^b = \begin{pmatrix} 0 & U'd\rho \wedge d\theta & (V\rho^{-1})'d\rho \wedge dz \\ -U'd\rho \wedge d\theta & 0 & -UV\rho^{-1}d\theta \wedge dz \\ -(V\rho^{-1})'d\rho \wedge dz & UV\rho^{-1}d\theta \wedge dz & 0 \end{pmatrix} \tag{3.4}$$

For the static case equations (2.11)–(2.13) can be rewritten, excluding π^a, as

$$T^a = \varepsilon^a_{bc} A^b_0 \, \underline{*}(\underline{D}A^c_0) - \varepsilon^a_{bc}\phi^b \, \underline{*}(\underline{D}\phi^c) \tag{3.5}$$

$$\underline{D} \, \underline{*D}A^a_0 = 0 \tag{3.6}$$

$$\underline{D} \, \underline{*D}\phi^a = 0 \tag{3.7}$$

respectively. The cylindrically symmetric ansatz for A^a_0 and ϕ^a reads

$$A^a_0 = \frac{1}{\rho}\begin{pmatrix} W(\rho) \\ 0 \\ 0 \end{pmatrix}, \quad \phi^a = \frac{1}{\rho}\begin{pmatrix} H(\rho) \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

The Hodge duality operation defined by the flat metric (3.1) can be summarized in matrix notation as

$$*\begin{pmatrix} d\rho \\ d\theta \\ dz \end{pmatrix} = \begin{pmatrix} \rho d\theta \wedge dz \\ \rho^{-1}dz \wedge d\rho \\ \rho d\rho \wedge d\theta \end{pmatrix} \tag{3.9}$$

Substituting (3.8), (3.2), and (3.9) into (3.6) and (3.7), we find

$$\rho^2 W'' - \rho W' = W(U^2 + V^2 - 1) \tag{3.10}$$

$$\rho^2 H'' - \rho H' = H(U^2 + V^2 - 1) \tag{3.11}$$

Using (3.3), (3.8), and (3.9), one transforms the torsion ("Cartan") equation (3.5) and the Einstein equation (2.10) to

$$\rho^2 U'' - \rho U' = U(V^2 + H^2 - W^2) \tag{3.12}$$

$$\rho^2 V'' - \rho V' = V(U^2 + H^2 - W^2 - 1) \tag{3.13}$$

These equations (3.10)–(3.13) form a closed system of second-order nonlinear equations for the functions U, V, W, H. The effective metric coefficients A, B, C are constructed from them via $A = UV\rho^{-2}$, $B = -\rho(V/\rho)'$, $C = -U'/\rho$.

4. EXACT SOLUTIONS WITH CONFINING PROPERTIES

Noticing that the general structure of the system (3.10)–(3.13) is close to that of the spherically symmetric problem in the Prasad–Sommerfield (1975) limit we look for the general nondegenerate solution for V, H, W in the form

$$V = K(\rho), \quad H = K(\rho)\cosh\gamma, \quad W = K(\rho)\sinh\gamma \tag{4.1}$$

where γ is an arbitrary constant. Using (4.1), one reduces (3.10)–(3.13) to the following system of coupled equations for two unknown functions K, U:

$$\rho^2 K'' - \rho K' = K(K^2 + U^2 - 1) \tag{4.2}$$

$$\rho^2 U'' - \rho U' = 2UK^2 \tag{4.3}$$

One immediately notices some resemblance of these equations to the spherically symmetric equations in the Prasad–Sommerfeld limit. However, the second terms on the l.h.s. produce an essential difference. In particular, the analysis of the power series expansions at zero and at infinity shows that the system (4.2)–(4.3) does not admit analytical solutions in which the function K goes to ± 1 at zero or at infinity. [To check our conclusions, we have used the REDUCE-based (Stauffer *et al.*, 1993) computer algebra system GRG in calculations (Zhytnikov *et al.*, 1992; Zhytnikov, 1991).

Integration of (4.2)–(4.3) is simplified greatly when one notices that this system is a consequence of the *first-order* system,

$$\rho K' - K = \varepsilon K U \tag{4.4}$$

$$\rho U' = \varepsilon K^2 \tag{4.5}$$

where $\varepsilon^2 = 1$. As one can see, (4.4)–(4.5) possesses a first integral

$$(\varepsilon U + 1)^2 - K^2 = C$$

with the help of which the final integration of (4.5) or (4.4) is straightforward. We have three cases, depending on the value of the constant C.

For $C = 0$ we find

$$K = \frac{\varepsilon_1}{\log(\rho/\rho_0)}, \quad U = \varepsilon_2\left(1 + \frac{1}{\log(\rho/\rho_0)}\right) \tag{4.6}$$

where $\varepsilon_{1,2}^2 = 1$, and ρ_0 is an integration constant.

For negative constant $C = -n^2$ we get

$$K_{(n)} = \frac{\varepsilon_1 n}{\cos[n \log (\rho/\rho_0)]}, \qquad U_{(n)} = \varepsilon_2 \left\{ -1 + n \tan \left[n \log \left(\frac{\rho}{\rho_0} \right) \right] \right\} \tag{4.7}$$

while for positive constant $C = n^2$ one finds

$$K_{(n)} = \frac{2\varepsilon_1 n(\rho/\rho_0)^n}{(\rho/\rho_0)^{2n} - 1}, \qquad = U_{(n)} = \varepsilon_2 \left(1 + n + \frac{2n}{(\rho/\rho_0)^{2n} - 1} \right) \tag{4.8}$$

Below for definiteness we will consider the plus signs $\varepsilon_1 = \varepsilon_2 = +1$ in these solutions.

The classical solutions (4.7) and (4.8) are parametrized by the two *real* constants: n and ρ_0. In fact, also any complex n is formally admissible and a complex integration constant ρ_0, but then the $SU(2)$ gauge fields are also complex and their physical interpretation is unclear. Solutions (4.8) with noninteger n are not analytical at zero and infinity. Negative n does not give anything new; it is easy to see that

$$K_{(-n)} = K_{(n)}, \qquad U_{(-n)} = U_{(n)}$$

Trivial $n = 0$ gives also a solution, which, however, does not reduce to (4.6).

All the solutions are singular. The fields (4.7) have an infinite number of singular points, while the configurations (4.6) have logarithmic and (4.8) a simple pole behavior at a point $\rho = \rho_0$,

$$K_{(n)}|_{\rho\to\rho_0} = U_{(n)}|_{\rho\to\rho_0} = \frac{\rho_0}{\rho - \rho_0} \tag{4.9}$$

This is most easily seen for integer n; then one can use the explicit formula $(r^{2n} - 1) = (r - 1)(r^{2n-1} + r^{2n-2} + \ldots + r + 1)$ (with $r = \rho/\rho_0$).But (4.9) is valid also for arbitrary noninteger n. This is the same singularity which is typical for spherically symmetric solutions with confining properties (Lunev 1993b; Singleton, 1995). However, unlike the spherical case, the Yang–Mills potentials and scalar fields (3.2), (3.8) are regular at the origin $\rho = 0$. The rest of the paper is devoted to the discussion of the solutions (4.8).

Without loss of generality, we will put $\rho_0 = 1$ in our subsequent study of the solution properties. Let us analyze the particular solution with $n = 1$ in greater detail.

The effective "gravitational" fields, which completely characterize the Yang–Mills field configuration, are calculated straightforwardly. When $n =$ 1 we find for the torsion and curvature the following expressions, respectively,

$$T^a = \frac{-8\rho}{(\rho^2 - 1)^3} \begin{pmatrix} 0 \\ d\rho \wedge d\theta \\ d\rho \wedge dz \end{pmatrix} \tag{4.10}$$

$$R_a^b = \frac{-4\rho}{(\rho^2 - 1)^2} \begin{pmatrix} 0 & d\rho \wedge d\theta & d\rho \wedge dz \\ -d\rho \wedge d\theta & 0 & \rho d\theta \wedge dz \\ -d\rho \wedge dz & -\rho d\theta \wedge dz & 0 \end{pmatrix} \tag{4.11}$$

Recall that the Riemann–Cartan curvature (4.11) is in fact the magnetic part of the Yang–Mills field strength.

The effective three-metric components A, B, C are also easily calculated to give for the line element

$$ds^2_{\text{eff}} = \frac{16}{(\rho^2 - 1)^4} (\rho^2 \, d\rho^2 + \rho^4 \, d\theta^2 + dz^2) \tag{4.12}$$

As we see, all the fields are zero at the symmetry axis, become infinite at a distance $\rho = 1$, and then rapidly fall off to zero at large distances from the axis. Solutions with $n > 1$ have the same general properties, only they decrease more quickly when $\rho \to 0$ and when $\rho \to \infty$. The physical interpretation of such a Yang–Mills configuration is clear: A region of space inside the tube of the radius $\rho = 1$ is separated by an infinite potential barrier from the outside space, providing a classical confinement of any matter with gauge charges in a stringlike structure. The absence of the gauge field at $\rho = 0$ is a classical counterpart of asymptotic freedom, since no force is acting on the gauge charges on the axis. One thus may call the family (4.8) confining string solutions. [Similar solutions in the Euclidean self-dual Yang–Mills theory were previously discussed by Saclioglu (1981, 1984), without, however, a study of confining properties.]

Due to the simple pole singularity (4.9) the total energy of all the solutions diverges, and it seems necessary to introduce a proper cutoff as with the formally infinite-energy Coulomb solution, following the suggestion of Singleton (1995). It should be stressed, however, that the confining property of the classical solutions (4.8) is a direct consequence of their singularity.

5. QUANTUM PARTICLE CONFINEMENT

Analogously to the spherically symmetric case (Lunev, 1993; Singleton and Yoshida, 1995), one can demonstrate that quantum particles with gauge charge are indeed confined inside the cylindrical domain near the symmetry axis. Let us consider the isospin-1/2 scalar particle in an external gauge field (4.8). For simplicity we will choose $\gamma = 0$ in (4.1) (thus eliminating the time component of the gauge field), and $n = 1$, and set the integration constant $\rho_0 = 1$ (for other values of n and ρ_0 the results are qualitatively the same).

The dynamics of a 2-component scalar field Ψ^A, $A = 1, 2$, with mass M in the external gauge field is described by the Klein–Gordon equation

$$\ddot{\Psi}^A - \underline{*}D\, \underline{*D}\Psi^A + M^2\Psi^A = 0 \tag{5.1}$$

where the covariant derivative is

$$D^A_B := d\delta^A_B - \frac{i}{2}A^a\, (\sigma_a)^A_B$$

and σ_a are 2×2 Pauli matrices.

Substituting the $n = 1$ (4.8) gauge field configuration into (5.1), and looking for the finite-energy solutions

$$\Psi^A = e^{-itE}\psi^A(\rho, \theta, z)$$

we find

$$-\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi^A}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi^A}{\partial\theta^2} + \frac{\partial^2\psi^A}{\partial z^2}\right)$$
$$+\left(M^2 - E^2 + \frac{\rho^2 + 1}{(\rho^2 - 1)^2} - \frac{2}{(\rho^2 - 1)}\Lambda\right)\psi^A = 0 \tag{5.2}$$

where we denote a linear differential operator

$$\Lambda := -i\left(\sigma^2\frac{\partial}{\partial z} - \sigma^3\frac{\partial}{\partial\theta}\right) \tag{5.3}$$

It is easy to see that the operators

$$-i\frac{\partial}{\partial\theta}, \qquad -i\frac{\partial}{\partial z}, \qquad \Lambda$$

commute with each other and with the differential operator in (5.2). Hence one can look for the solution which is a common eigenstate of all the operators, and separate variables as

$$\psi^A = \frac{1}{\sqrt{\rho}}\varphi_{ml\lambda}(\rho)e^{imz+il\theta}\psi^A_\lambda \tag{5.4}$$

where constants m and l are the eigenvalues of $-i\partial_z$ and $-i\partial_\theta$, respectively, and ψ^A_λ is the eigenfunction

$$\Lambda\psi^A_\lambda = \lambda\psi^A_\lambda$$

with the eigenvalues $\lambda = \pm\sqrt{m^2 + l^2}$.

Substituting (5.4) into (5.2), we find for the radial function the one-dimensional stationary Schrödinger equation

$$\left(-\frac{d^2}{d\rho^2} + V(\rho) \right) \varphi_{ml\lambda} = \mu^2 \varphi_{ml\lambda} \tag{5.5}$$

with the potential

$$V(\rho) = \frac{\rho^2 + 1}{(\rho^2 - 1)^2} - \frac{2\lambda}{\rho^2 - 1} - \frac{l^2 + 1/4}{\rho^2} \tag{5.6}$$

and eigenvalue

$$\mu^2 = E^2 - M^2 + m^2$$

The infinite potential barrier at $\rho = 1$ with the leading term $V \sim 2/(\rho - 1)^2$ is known to be completely impenetrable (Dittrich and Exner 1985) [the exact form of the eigenfunctions in (5.5) is not important]. Thus all quantum particles with a gauge charge cannot move out of the "thick string" region $0 \leq \rho \leq 1$, providing a picture of confinement. The same conclusion is valid also for the Dirac spinor particles.

It seems worthwhile to mention that the result obtained is nontrivial because not every singular potential (infinitely high barrier) is necessarily impenetrable. In particular, potentials $a/(\rho - 1)^2$ are *not confining* for $a \leq 1$; see Dittrich and Exner (1985).

6. DISCUSSION AND CONCLUSION

In this paper we obtained a family of new exact cylindrically symmetric solutions for the $SU(2)$ gauge Yang–Mills theory. Like the earlier reported spherically symmetric solutions, these are also singular and can thus provide a mechanism for classical confinement. This is another demonstration of the fruitfulness of deriving analogies and constructing direct mappings between the gauge theories of internal symmetry groups and gravity theory.

It seems necessary to mention certain problems and prospects for the results obtained. First of all, QCD works not with $SU(2)$ group, but with $SU(3)$. Straightforward analysis shows that there exist, a generalization of the above results to the $SU(3)$ case, which arise from the embedding of $SU(2)$ into $SU(3)$. However, the geometrical meaning of the mapping between the gauge model and gravity is at the moment unclear, although one can point to the work of Bauer *et al.* (1994), Bauer and Freedman (1995) and Lunev (1996) where attempts were made to prove the existence of the reasonable generalization of the mapping between $SU(N)$, $N \geq 3$, gauge theory and gravitation, thus demonstrating that the coincidence of the $SU(2)$ group dimension with the number of spatial coordinates is accidental and plays no decisive role. The arising effective spatial geometry is necessarily non-Riemannian in this case. Another point which deserves special attention is the physical interpretation of the infinite energy of the confining solutions. As we already mentioned, finite-energy configurations cannot provide the binding of gauge charges. However, a regularization similar to the one which is applied to the (originally infinite-energy) meron solutions might be necessary in our case. Work is in progress in this direction.

A possible physical importance of the new solutions would be a development of the toroidal bag constituent model for the hadrons. Recently an analogous spherical bag model based on classical singular solutions was discussed in Singleton and Yoshida (1995). The toroidal glueballs were analyzed, e.g., in Robinson (1980), and later Mahajan and Valanju (1987a, b) made an attempt to reformulate this model within the cylindrically symmetric approximation to the Yang–Mills toroidal configurations. Such an approximation arises naturally from an exact cylindrical solutions by imposing a periodicity condition on the z coordinate. Some remarks are in order about Mahajan and Valanju (1987a, b). Using an ansatz similar to but somewhat different from (3.2) for the cylindrically symmetric Yang–Mills field, they failed to describe an exact classical solution. They, however, noticed the possibility of a simple pole singularity of the type (4.9) and tried to avoid it by assuming that the integration constant analogous to ρ_0 is complex, at the same time speculating that a smooth matching of such a solution with the real analytical power series solutions at zero and infinity exists. As is clearly shown in the present paper, no such matching exists and the only possibility to avoid the singularity for real ρ is to make a solution complex on all the ρ axis. Such a complex solution, although formally admissible, most probably will be unphysical. Our results thus provide corrections and generalization of the Mahajan and Valanju (1987a, b).

It certainly may turn out that the true explanation of the QCD confinement involves purely quantum arguments which are unrelated to the speculative, to an extent, calculations based on the new spherically symmetric (Lunev, 1993b; Singleton, 1995) and cylindrically symmetric (4.8) Yang–Mills solutions. Nevertheless, it seems worthwhile to notice once again the power and flexibility of the classical Yang–Mills theory, which provides alternative geometrical mechanisms for understanding, maybe at least partly, such intriguing problems as confinement.

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